# Assignment 5.

This homework is due *Thursday*, October 4.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 7.

## 1. Quick reminder

The (Lebesgue) outer measure is a function $m^* \colon \{\text{subsets of } \mathbb{R}\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined as

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; I_1, I_2, \dots \text{ open bounded intervals}, A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

The outer measure $m^*$ has the following properties:

- $m^*(I) = \ell(I)$ for every interval $I$.
- $m^*$ is monotone: $m^*(A) \leq m^*(B)$ if $A \subseteq B$.
- $m^*$ is translation invariant: for any $A \subseteq \mathbb{R}$, for any $y \in \mathbb{R}$,
$$m^*(A + y) = m^*(A).$$
- $m^*$ is countably subadditive:
$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*(A_k).$$

## 2. Exercises

(1) (2.2.8+) Let $B$ be the set of rational numbers in the interval $[0, 1]$. Let $\ell(I)$ denote length of an interval $I$.
  (a) Let $\{I_k\}$, $k = 1, 2, \dots, n$, be a finite collection of open intervals that covers $B$. Prove that $\sum_{k=1}^{n} \ell(I_k) \geq 1$.
  (b) Show that for every $\varepsilon > 0$, there is a countable open cover $\{I_k\}$, $k = 1, 2, \dots$, of $B$ such that $\sum_{k=1}^{\infty} \ell(I_k) \leq \varepsilon$.

  Comment: this exercise shows that infinite covers may be very different from finite ones.

(2) (2.2.9) Prove directly from definition of $m^*$ that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

(3) (2.2.10) Suppose $A$ and $B$ are *separated* bounded sets, i.e. there is an $\alpha \in \mathbb{R}$, $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A$, $b \in B$. Prove directly from definition of $m^*$ that $m^*(A \cup B) = m^*(A) + m^*(B)$.

(4) Prove that a countable union of set of (Lebesgue) measure 0 is a set of measure 0:
  (a) directly from definition of $m^*$,
  (b) using subadditivity of $m^*$.

— see next page —

(5) (a) Define Jordan outer measure as a function $j^* : \{\text{subsets of } \mathbb{R}\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ as

$$j^*(A) = \inf \left\{ \sum_{k=1}^{n} \ell(I_k) \;\middle|\; I_1, \ldots, I_n \text{ open intervals}, A \subseteq \bigcup_{k=1}^{n} I_k \right\}.$$

(That is, the same definition as $m^*$, except that only finite open covers are allowed.)
Prove that $j^*(\emptyset) = 0$, $j^*$ is monotone, *finitely subadditive*, but *not countably* subadditive. (*Hint:* For the last part, see problem 1a.)

(b) Define function $c^* : \{\text{subsets of } \mathbb{R}\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ as

$$c^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; I_1, I_2, \ldots \text{ closed bounded intervals}, A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

(That is, the same definition as $m^*$, except that closed intervals covers are used instead of open ones.)
Prove that $c^* = m^*$.
(*Hint:* Prove that every closed intervals cover can itself be covered by open intervals with arbitrarily small "surplus", and vice versa.)

## 3. Extra exercise

(6) Prove that the Lebesgue outer measure $m^*$ is not *continuous*, i.e. that it is not always true that

$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n\to\infty} m^*\left(\bigcup_{k=1}^{n} A_k\right).$$